

ALTERNATING AUGMENTATIONS OF LINKS

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ABSTRACT. We show that one can interweave an unknot into any non-alternating connected projection of a link so that the resulting augmented projection is alternating.

In this paper, we will use the term link to mean a tame link embedded in S^3 . A link projection D is the image of a link under a regular projection into S^2 . D is a finite four-valent graph in S^2 . If D is connected then each complement component is a disk; we call the closure of one of these disks a region. Define a labelling of the edges of D in the manner of Fig. 1. Every edge of D receives two labels, one corresponding to each end. An alternating edge is labelled with a plus and a minus while a non-alternating edge receives two pluses or two minuses. Hence, an alternating projection is one in which every edge is labelled with a plus and a minus. We call a non-alternating edge labelled $++$ a positive non-alternating edge and a non-alternating edge labelled $--$ a negative non-alternating edge. Note that a labelled knot projection is equivalent to the usual knot diagram only with the crossing information stored as edge labels. In the spirit of Adams[1], we define an augmented link projection of D to be the union of D with an unlink that projects to disjoint simple closed curves in S^2 .

We begin with a proposition originally outlined by Thistlethwaite[2].

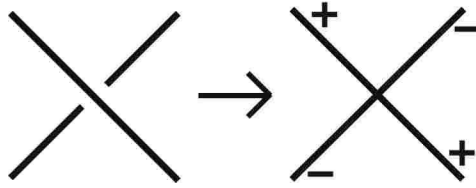


FIGURE 1.

Proposition 1. *Any non-alternating connected link projection can be augmented so that it becomes alternating.*

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Proof. Choose a region R of the non-alternating projection D . Let n be the number of positive non-alternating edges of R . Since each vertex of ∂R contributes one $+$ and one $-$ label to the edges of ∂R and each edge receives exactly two labels, then ∂R contains exactly n negative non-alternating edges. Let α be a non-alternating edge of ∂R . After choosing an orientation of ∂R and again counting the labels contributed by the vertices, the sign of the next non-alternating edge in the direction of the orientation must be opposite that of α . The slogan here is that the sign of the non-alternating edges of ∂R alternate.

With this knowledge, we can construct an augmented alternating projection G from D as follows. Introduce vertices into the interiors of all non-alternating edges of D . In every non-alternating region R , we join pairs of such points together by edges which are disjoint, lie in the interior of R and join consecutive non-alternating edges as depicted in Fig. 2. We call these new edges augmenting edges. Since the augmenting edges connect non-alternating edges of opposite sign then there is a consistent alternating labelling of the edges of G . In particular, the end of an augmenting edge which bisects a positive non-alternating edge of D receives a $+$ label; the end that bisects a negative non-alternating edge receives a $-$ label. Since the augmenting edges never cross themselves, the closure of $G-D$ ($cl(G-D)$) is a disjoint collection of simple closed curves embedded in S^2 . Because this process converts every non-alternating edge of D into two alternating edges of G by interweaving alternating unknots into D , G is an alternating augmented projection of D . \square

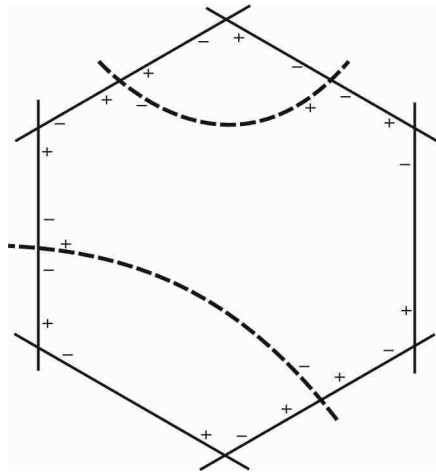


FIGURE 2.

Figure 3 illustrates an operation on link projections we call a Type I move.

Lemma 2. *Given an alternating connected projection of a link, a Type I move results in an alternating link projection.*

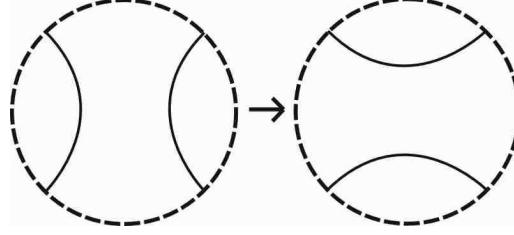


FIGURE 3. Type I Move

Proof. Let D be the link projection and α and β be the edges involved in the Type I move. α and β are boundary edges of a common region R . Since D is alternating, all edges of R are labelled with both a plus and a minus. Given two consecutive edges of ∂R their shared vertex contributes a plus label to one edge and a minus label to the other. Thus, a choice of label for a single edge determines the label of all the edges of ∂R . In this way, an alternating label for α determines the label for β , giving rise to the following two possibilities.

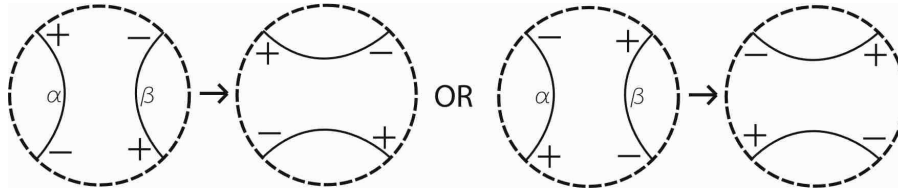


FIGURE 4.

In each case, the type I move preserves alternation. \square

Figure 5 illustrates an operation on link projections we call a Type II move.

Lemma 3. *Given an alternating connected projection of a link, a Type II move (after choosing labels incident to the new vertices) results in an alternating link projection.*

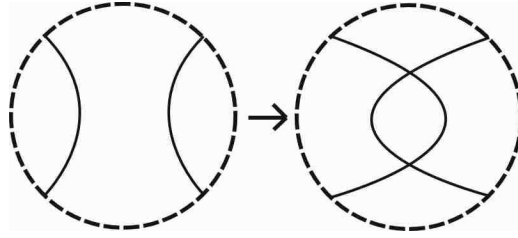


FIGURE 5. Type II Move

Proof. Let D be the link projection and α and β be the edges involved in the Type II move. We again use the fact that α and β are boundary edges to a common region in D to deduce that an alternating label for α determines the label for β . Hence, we have only the two following possibilities for the labels of α and β .

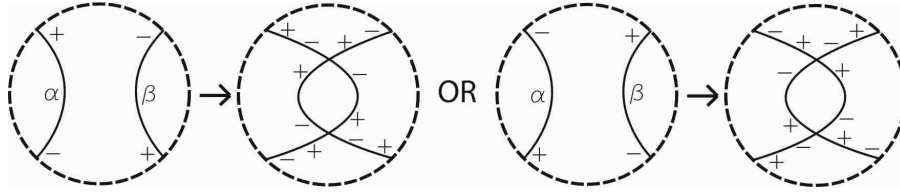


FIGURE 6.

In each case, as shown in Figure 6, we may choose a labelling of the ends of the edges incident to the new vertices so that the resulting diagrams are alternating. \square

Theorem 4. *Given any connected projection of a non-alternating link, we can augment the projection by adding a single unknotted component so that the resulting link projection is alternating.*

Proof. Let D be a regular projection of a non-alternating link. Create G , an alternating augmented projection of D , as described in Prop.1. Let $cl(G - D) = \bigcup_{1 \leq i \leq n} C_i$ where each C_i is a simple closed curve in S^2 . If $n = 1$ then there is nothing to prove. If $n \geq 2$ then there is a path component A of $S^2 - cl(G - D)$ whose closure has at least two distinct boundary components C_i and C_j .

If C_i and C_j contain boundary edges of a common region in G then we may use a type I move to join C_i and C_j into a single simple closed curve in S^2 . By Lemma 2 the resulting projection is alternating.

If C_i and C_j do not contain boundary edges of a common region in G , then consider a path μ in A transverse to G so that $\partial\mu = \{a, b\}$ for $a \in C_i$ and $b \in C_j$. We can propagate C_i along μ using type II moves

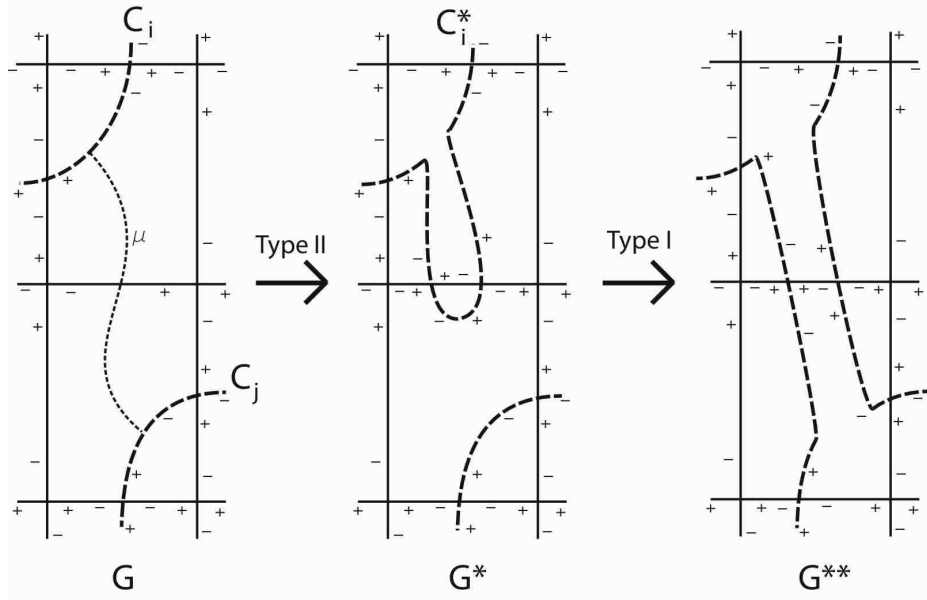


FIGURE 7.

as depicted in Fig. 7 until C_i^* (the image of C_i under type II moves) and C_j contain boundary components of a common region R . Call this projection G^* . Since a type II move is an isotopy of C_i in S^2 and μ was restricted to A , C_i^* is a simple closed curve which does not intersect the other C_j . Hence, G^* is an augmented link of D . By Lemma 3, G is alternating implies we may choose labels at the new vertices so that G^* is alternating. We then use the type I move to connect sum the disjoint simple closed curves C_i^* and C_j into a single simple closed curve. Call the resulting projection G^{**} . Since G^* is alternating so is G^{**} , by Lemma 2. Hence, G^{**} is an alternating augmented link of D with one less augmenting component than G . Repeat this process until there is an alternating augmented projection of D with exactly one augmenting component, proving the theorem. \square

REFERENCES

- [1] C. Adams, "Augmented alternating link complements are hyperbolic," *London Mathematical Society Lecture Notes Series, 112: Low Dimensional Topology and Kleinian Groups*, pp.115-130, Cambridge University Press, Cambridge, 1986.
- [2] Thistlethwaite, Morwen B, "An upper bound for the breadth of the Jones polynomial," *Math. Proc. Cambridge Philos. Soc.* 103 (1988), no. 3, 451–456.